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Philip Saffman and viscous flow theory

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Philip Saffman made valuable theoretical contributions to different areas of low-Reynolds-number hydrodynamics. Three themes are selected for discussion here: (i) the lift force on a sphere in a shear flow at small, but finite Reynolds number, (ii) Brownian motion in thin liquid films, and (iii) particle motion in rapidly rotating flows. In addition, brief descriptions are given of some of Saffman's other contributions including dispersion in porous media, the average velocity of sedimenting suspensions, and compressible low-Reynolds-number flows.

1. Introduction

Philip Saffman made many outstanding and original contributions to fluid dynamics during his research career. A glance at his list of publications shows that in the twenty year period 1956–1976, he studied, sometimes in collaboration and other times by himself, a wide variety of viscous flow problems. Arguably the most well-known of these research investigations is the phenomenon of viscous fingering, i.e. the Saffman–Taylor instability. Many of Saffman's other studies of low-Reynolds-number flows seem to be much less widely known and the main goal of this paper is to introduce the reader to these different investigations, their principal conclusions, and some of the modern extensions.

By way of introduction, Saffman published three articles in volume 1 of the *Journal of Fluid Mechanics*. Two of these papers concerned particle dynamics at low Reynolds numbers. The first examined the rise of small air bubbles (Saffman 1956*a*) and presented both experiments and theory to characterize the change in the bubble's trajectory from a vertical straight line to a zig-zag path (motion confined to a vertical plane) or to a spiral motion. The reader may recall similar phenomena while observing bubbles in champagne. This research topic still has open questions concerning the detailed dynamics of the zig-zag and spiral trajectories, as well as the transition from one to the other, and is an active area of investigation, e.g. see the recent review by Magnaudet & Eames (1999). The second of these papers (Saffman 1956*b*) concerned the orientational motion of nearly spherical particles in a simple shear flow. As described by Batchelor (1976), G. B. Jeffery (1922) had originally shown that, in a zero-Reynolds-number flow, there is a one-parameter family of possible orbits for the orientation of a spheroidal particle in a shear flow (the so-called Jeffrey orbits), and suggested that, regardless of initial conditions, the actual orbit would correspond to a minimum in the energy dissipation; for example, in a shear flow a prolate spheroid would align with its axis parallel to the vorticity vector. Although experiments by G. I. Taylor (1923) qualitatively confirmed these results, Saffman's calculation accounting for the first influences of inertia demonstrated that the experimental observations occurred too quickly for inertia to be responsible. In fact, non-Newtonian influences are thought to be important for interpreting Taylor's experiments (Batchelor 1976) and, in a Newtonian fluid, particles adopt an orientation where the energy dissipation is a maximum (Harper & Chang 1968). Hence, when just beginning his research career, Saffman made two significant contributions to viscous flow theory, both of which remain significant today. There were many more to come!

Professor Saffman made important contributions to such distinct problems as (i) the lift force on particles when the Reynolds number is small and pointed the way toward explaining the original experimental observations of Segré & Silberberg (1962*a, b*), (ii) a hydrodynamic description of the diffusion of proteins in bilayer membranes, which illuminated experiments being performed in biology laboratories (as well as transport processes in our own cells), and (iii) theories for particle motion in rapidly rotating flows. Other distinct areas of low-Reynolds-number hydrodynamics that bear Saffman's footprint include the mechanics of sedimenting suspensions and compressibility effects in low-Reynolds-number (lubrication) flows. Saffman also made an important contribution to the description of dispersion accompanying flow in a porous medium. Each of these, probably less well-known, research contributions is an important publication that focused on significant, poorly understood scientific questions and provided answers that remain of fundamental importance and interest today. Not surprisingly, each of the papers in these different research areas combine physical insight and asymptotic scaling analyses that are the hallmark of great theoretical pioneering studies and are characteristic of Saffman's scientific papers.

In this article I shall try to summarize these less well-known investigations and put Saffman's theoretical work in a wider context. The discussion will focus on (i)–(iii) above, as well as mentioning recent related studies. Emphasis will be given to theoretical work, since this is most closely related to Saffman's contributions. It is not possible to go into great detail but I will attempt to highlight the important results by way of scaling arguments. In such a short survey of these different subjects it is also inevitable that significant extensions will be either treated too tersely, or even worse, not mentioned at all. In any event, I hope that the spirit and style of Saffman's fundamental, and still valuable, contributions to low-Reynolds-number hydrodynamics comes through.

2. Lateral migration: lift in low-Reynolds-number flows

Synopsis: Saffman's contribution to this problem involved understanding how inertia of the fluid leads to a lift force on particles suspended in flow and the subsequent drift of the particles across streamlines. The required calculation utilizes singular perturbation methods and the predictions have been shown to be in good agreement with experiment.

Suspension flows are common and situations where the particle-scale Reynolds numbers are small arise in many dispersed two-phase flows with viscous continuous phases or small suspended particles (e.g. blood flow). In the zero-Reynolds-number limit, Stokes determined the hydrodynamic (drag) force on a steadily translating spherical particle and obtained the familiar result that the hydrodynamic force acting on the particle is $\boldsymbol{F} = -6\pi\mu a \boldsymbol{U}_p$, where μ is the fluid viscosity, a the particle radius and $\boldsymbol{U}_p$ the translational velocity relative to the surrounding fluid. There are at least four main lines of research associated with extending this result, including (1) accounting for the influence of boundaries, (2) accounting for the influence of fluid and particle inertia, (3) accounting for the influence of nearby particles or a suspension, and (4) allowing for non-spherical shapes. In this section we discuss the influence of fluid inertia, which can give rise to *lift* forces, i.e. forces perpendicular to the direction of translation. The effect of lift is to produce drift of particles relative to the streamline passing through the particle centre, which can be significant for establishing the distribution of particles in a given flow.

2.1. *Background*

In a very viscous fluid a particle that experiences a net force (e.g. a particle with different density to the surrounding fluid) moves steadily relative to the fluid. Also, owing to its finite size, a neutrally buoyant particle moves relative to the local fluid if there is a pressure gradient across the particle, as occurs for example in a parabolic channel or pipe flow. However, when the undisturbed motion is zero Reynolds number and unidirectional, the force on a spherical particle is collinear with the flow direction and no 'lift', or sideways, force is possible, as explained originally by Brenner & Happel (1958), Bretherton (1962) and Saffman (1965). Thus, for pressure-driven flow in a tube or channel, spherical particles are predicted to not drift across streamlines. However, experiments by Segré & Silberberg (1962*a*, *b*) showed instead that neutrally buoyant particles in laminar pipe flow migrated across streamlines and tended to adopt a radial position approximately $0.6R_t$ from the tube centreline, where R_t is the tube radius. This rather dramatic result, often referred to as the tubular pinch effect, has been reproduced in other experiments (e.g. Tachibana 1973; Aoki, Kurosaki & Anzai 1979; Berge 1990; see also Walz & Grün 1973), and has been studied for both neutrally buoyant and non-neutrally buoyant particles. Although Saffman's theoretical contribution involved calculating the influence of fluid inertia on particle motion, he also was the first to understand the crucial importance of walls for explaining the experiments of Segré & Silberberg. To quote Saffman (1965, p. 385):

> The full problem is one of great difficulty, as not only is the effect of inertia to be calculated for a particle in a parabolic velocity profile, but also the presence of the tube walls must be taken into account. The walls are clearly all important to the existence of the phenomenon, if only because without walls the particle would never know (so to speak) when it was the appropriate distance from the axis.

Saffman (1965) provided the first detailed analysis illustrating the manner in which fluid inertia can lead, in the presence of a local shear flow and the particle moving relative to the fluid, to a cross-stream migration (he did not account for the wall interaction nor treat the neutrally buoyant limit of most direct relevance to the Segré–Silberberg experiments). More importantly, the theoretical approach taken by Saffman (see also Childress 1964) for analysing this flow is very fruitful and has been adapted by many other workers, e.g. Lovalenti & Brady (1993) used similar ideas to treat transient particle motion at small but finite Reynolds numbers and Legendre & Magnaudet (1997) used Saffman's approach to treat the lift forces on spherical bubbles and drops. The migration problem for both neutrally buoyant and non-neutrally buoyant particles has now been well-studied theoretically and compared (rather successfully) with experiment and applied to other suspension flows beyond those envisioned originally. A review of research up to 1979 on the influence of fluid inertia on particle migration is provided by Leal (1980) and more recent research is summarized and extended by Hogg (1994). In particular, Hogg treats particle motion in Poiseuille channel flow, considers both horizontal and vertical channels, and clearly delineates different regimes of shear, non-neutrally buoyant particles, and boundary influences.

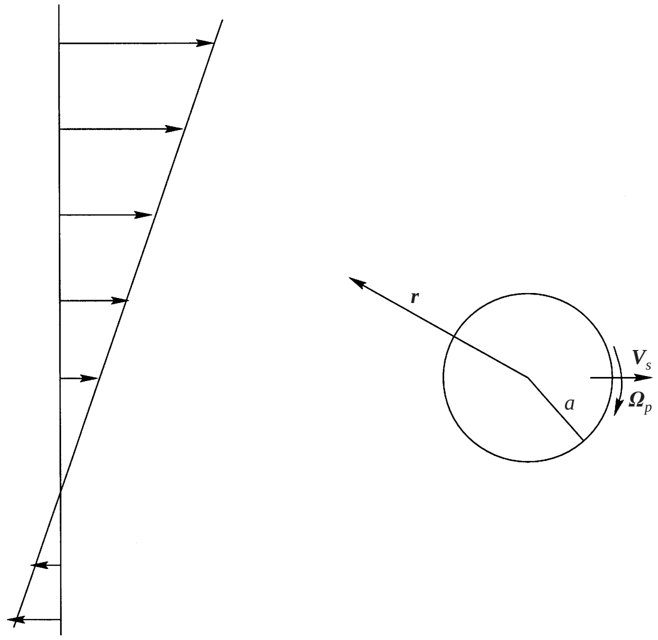

FIGURE 1. Motion of a particle relative to a locally linear flow.

Before introducing some mathematical details we note that in general there are at least two small parameters in this problem: the particle Reynolds number and the dimensionless distance of the particle from the nearest boundary. The analysis can be carried out via a regular perturbation expansion when the boundaries lie closer to the particle than the (large) Oseen distance at which fluid inertia has a comparable magnitude to viscous effects (Cox & Brenner 1968; Ho & Leal 1974). In the opposite limit where fluid inertia cannot be neglected at large distances from the particle, the analysis requires singular perturbation methods, which were developed for the problem at hand by Saffman (1965). In fact, Saffman comments that the analysis has the same mathematical spirit as that described by Childress (1964) who considered inertial effects (the Coriolis acceleration) in particle motion in rotating fluids (see § 4).

2.2. *Mathematical description*

In order to describe the low-Reynolds-number lift calculation and to focus on the most important mathematical ideas we denote the fluid velocity by $\boldsymbol{u}$ and consider a spherical particle with radius a translating at velocity $\boldsymbol{U}_p$ and rotating at angular velocity Ω_p in a specified undisturbed flow $\boldsymbol{u}^\infty$ (figure 1). It is convenient to express $\boldsymbol{u}^\infty$ relative to the undisturbed motion evaluated at the instantaneous centre of the particle $\boldsymbol{U}^\infty$, and so write $\boldsymbol{u}^\infty = \boldsymbol{U}^\infty + \boldsymbol{u}'^\infty$; Saffman considered the special case of a shear flow, which we shall denote $\boldsymbol{u}'^\infty = \boldsymbol{\Gamma} \cdot \boldsymbol{r}$, where $\boldsymbol{r}$ is a position vector measured relative to the particle centre. Further, it is convenient to express the fluid velocity in disturbance variables, $\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{u}^\infty$, measuring deviations from the imposed flow. The slip velocity of the particle relative to the fluid is defined as $\boldsymbol{V}_s = \boldsymbol{U}_p - \boldsymbol{U}^\infty$. The Navier–Stokes and continuity equations, written in terms of disturbance variables $\boldsymbol{u}'$ for a steady flow relative to a coordinate system fixed to the particle, are (e.g. Hogg 1994)

$$\rho\left(-\boldsymbol{V}_s \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}'^\infty \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla \boldsymbol{u}'^\infty + \boldsymbol{u}' \cdot \nabla \boldsymbol{u}'\right) = -\nabla p' + \mu \nabla^2 \boldsymbol{u}' \quad \text{and} \quad \nabla \cdot \boldsymbol{u}' = 0. \quad (1)$$

The corresponding boundary conditions are

$$\boldsymbol{u}' = \boldsymbol{V}_s + \Omega_p \wedge \boldsymbol{r} - \boldsymbol{u}'^{\infty} \quad \text{on } S_p, \quad \boldsymbol{u}' = \boldsymbol{0} \text{ on boundaries,} \quad \text{and} \quad \boldsymbol{u}' \to \boldsymbol{0} \text{ at } \infty. \tag{2}$$

For the special case of a homogeneous shear flow studied by Saffman, equation (1) is

$$\rho \left(-\boldsymbol{V}_s \cdot \nabla \boldsymbol{u}' + (\Gamma \cdot \boldsymbol{r}) \cdot \nabla \boldsymbol{u}' + \Gamma \cdot \boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla \boldsymbol{u}'\right) = -\nabla p' + \mu \nabla^2 \boldsymbol{u}'. \tag{3}$$

Now, if inertia is neglected and the undisturbed flow is unidirectional, these equations are incapable of predicting cross-stream migration of a spherical particle. With small but finite fluid inertia the particle experiences a lift force $\boldsymbol{F}_L$ and the corresponding steady migration velocity $\boldsymbol{V}_L$ follows from $\boldsymbol{V}_L = (6\pi\mu a)^{-1}\boldsymbol{F}_L$. The question, though, remains of what is $\boldsymbol{F}_L$? There are two particle-scale Reynolds numbers: $\mathscr{R}_s$ based upon the slip velocity $\boldsymbol{V}_s$, i.e. $\mathscr{R}_s = \rho V_s a/\mu$, and $\mathscr{R}_G$ based on the local shear rate G, i.e. $\mathscr{R}_G = \rho G a^2/\mu$. Saffman's original analysis considered the limits

$$\mathscr{R}_s \ll \mathscr{R}_G^{1/2} \ll 1, \tag{4}$$

which physically correspond to the small effects of fluid inertia first becoming comparable to viscous effects on a length scale $a\mathscr{R}_G^{-1/2} = \left(\nu/G\right)^{1/2}$; the more familiar Oseen length scale owing to inertia associated with the slip motion is comparable to viscous effects on a length scale $a\mathscr{R}_s^{-1} = \nu/V_s$, which is much larger than $a\mathscr{R}_G^{-1/2}$ according to (4). Boundary effects are neglected by Saffman and so must lie at a distance beyond $O(a\mathscr{R}_G^{-1/2})$.

Since the Reynolds numbers are small, it is in the far field where inertial effects are first significant and to analyse the fluid motion in this far field Saffman made the important observation that the particle can be replaced by a point particle and the equations linearized. Thus, assuming (4), the far field is studied according to (these equations could be made dimensionless, but for this order-of-magnitude discussion it seemed reasonable to keep the equations dimensional)

$$\rho \left((\Gamma \cdot \boldsymbol{r}) \cdot \nabla \boldsymbol{u}' + \Gamma \cdot \boldsymbol{u}'\right) = -\nabla p' + \mu \nabla^2 \boldsymbol{u}' + 6\pi\mu a \boldsymbol{V}_s \delta(r), \tag{5}$$

as a particle with a finite slip velocity exerts a force on the fluid $6\pi a\mu \boldsymbol{V}_s$. Balancing the inertial terms on the large length scale $r = O(a\mathscr{R}_G^{-1/2})$ with the force/volume shows $O(6\pi a\mu V_s/r^3) = O(\rho u' G)$, which produces a velocity field $u' = O(6\pi\mathscr{R}_G^{1/2} V_s)$. The velocity gradients associated with this inertially generated velocity field produce a correction to the force acting on the particle, the so-called lift force, $F_L = O(6\pi a\mu \mathscr{R}_{\mathscr{G}}{}^{1/2} V_s)$. In particular, for a simple shear flow, with the slip velocity parallel to the flow direction, Saffman's detailed calculation yielded the lift force (often called the Saffman lift)

$$F_L = 6.46 a\mu \mathscr{R}_G^{1/2} V_s \tag{6}$$

or a lift velocity $V_L = 0.343\mathscr{R}_G^{1/2} V_s$ directed across the undisturbed streamlines moving opposite to V_s.†

The implication of this result for particle motion in pipe flows is evident, at least for those situations where the parabolic part of the flow is insignificant. If the particle

† In fact, Saffman made a small algebra error in the lift calculation in his published paper; a factor of $4\pi^2$ was lost in evaluating the integral expression for the lift. In a communication to the author, Professor Saffman commented that this error "was found *experimentally* by I.D. Chang, who devised an ingenious experiment to measure the lift and found that there was a continuous discrepancy of about 40 between the predicted lift and the measured lift. Clearly, the 40 must be coming from a $4\pi^2$ in the theory, and knowing what to look for, Chang found the error."

moves faster than the local fluid ($V_s > 0$), then the particle moves away from the axis, while if the particle lags the local velocity ($V_s < 0$), then the particle moves toward the axis (e.g. Saffman 1965, p. 394). Experiments (Aoki *et al.* 1979) demonstrate that the equilibrium position attained in tube flow can be shifted towards the boundary if the particle exceeds the local flow or toward the axis if the particle lags the flow (the particle speed was adjusted by changing the density difference between the particle and the fluid). These experimental results are in agreement with the predictions originally made by Saffman.

Research since Saffman's paper: The influence of the slip-velocity-generated inertial flow ($\boldsymbol{V}_s \cdot \nabla \boldsymbol{u}'$ in (3)) was considered by McLaughlin (1991). Setting $\epsilon = \mathscr{R}_G^{1/2}/\mathscr{R}_s$, which represents the ratio of the first two terms on the left-hand side of (3), then the lift velocity has the form $V_L = 0.343\mathscr{R}_G^{1/2} V_s f(\epsilon)$ and McLaughlin calculated $f(\epsilon)$ numerically. One feature of the numerical results is that for $\mathscr{R}_G^{1/2}/\mathscr{R}_s < 0.22$ the direction of the lift velocity is reversed and the particle moves (very slowly) in the direction of the faster streamlines. The case of spherical drops and bubbles, rather than rigid spheres, was addressed recently by Legendre & Magnaudet (1997), who noted that the lift result was generated by far-field influences and hence the viscosity contrast between droplets μ_d and the continuous phase μ enters very simply. With $\beta = (3\mu_d + 2\mu)/(\mu_d + \mu)$, the drift speed has the form $V_L = 0.343\mathscr{R}_G^{1/2} V_s \beta^2 f(\epsilon)$. The dependence on β^2 means that the lift force on a clean gas bubble is smaller than that on a rigid sphere (translating at the same speed) by a factor $(2/3)^2$. McLaughlin (1993) extended the solid sphere results to account for the presence of a nearby planar boundary. The generalization of Saffman's calculation to three-dimensional bodies in simple shear flow has been given by Harper & Chang (1968).

The case of neutrally buoyant spheres corresponds to a stresslet forcing in the far field. This case was considered by Schonberg & Hinch (1989) using a theoretical analysis in the spirit of Saffman's calculation. The general case with both point force and stresslet forcings in the far field was described by Hogg (1994) and more recently by Asmolov (1999); note that there are some small numerical differences between some of the results reported by Hogg and Asmolov. With reference to (1) the far field is analysed according to

$$\rho\left(-\boldsymbol{V}_s \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}'^{\infty} \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla \boldsymbol{u}'^{\infty}\right) = -\nabla p' + \mu \nabla^2 \boldsymbol{u}' + 6\pi\mu a \boldsymbol{V}_s \delta(r) + \boldsymbol{S} \cdot \nabla \delta(r), \quad (7)$$

where $\boldsymbol{S}$ is the particle stresslet, which for a rigid sphere is given by $\boldsymbol{S} = (20\pi\mu a^3/3)\boldsymbol{E}^{\infty}$, with $\boldsymbol{E}^{\infty}$ the rate-of-strain tensor of the undisturbed flow. For the case of a neutrally buoyant particle, $\boldsymbol{V}_s = \boldsymbol{0}$, and the inertia-induced migration velocity has a magnitude established by balancing far-field inertia $O(\rho u' G)$ with the stresslet driving force/volume $O(\mu a^3 G/r^4)$ on a length scale $r = O(a\mathscr{R}_G^{-1/2})$. The force-free particles translate with this velocity, hence the migration speed is $O(aG\mathscr{R}_G)$. Schonberg & Hinch included the influence of walls and found that particles migrate across the channel to a Reynolds-number-dependent equilibrium distance between the centreline and wall with reasonable agreement between their channel-flow calculations and the original tube-flow experiments of Segré & Silberberg (1962*a*, *b*).

For the case of particle motions in channels and tubes, the calculations of Schonberg & Hinch (1989) (and Asmolov 1999) are to be distinguished from the calculations of Ho & Leal (1974). The former used singular perturbation methods to treat the case where the influence of fluid inertia was significant on length scales comparable to, or smaller than, the distance ℓ to the closest wall, i.e. $\ell \gg a\mathscr{R}_G^{-1/2}$, while the latter used regular perturbation methods to analyse the opposite limit, $\ell \ll a\mathscr{R}_G^{-1/2}$. It is thus convenient to organize the possible cases according to the magnitude of a channel Reynolds number, $\mathscr{R}_c = \mathscr{R}_G/(a/\ell)^2$, representing the (square of the) distance to the closest boundary relative to the (square of the) distance where inertial influences are important (e.g. Hogg 1994, table 1).

In addition, the lift calculation has been performed by generalizing the simple shear flow to an arbitrary linear flow, i.e. $\boldsymbol{u}'^{\infty} = \boldsymbol{\Gamma} \cdot \boldsymbol{r}$. This flow was considered by Miyazaki, Bedeaux & Avalos (1995) with specific results given for a simple shear flow. It is interesting to note that there appears to be some (reasonably small) numerical discrepancy between these latest results and those reported by Harper & Chang (1968) and Hogg (1994, Appendix E).

As a final recent application that utilizes some of the singular perturbation ideas pioneered by Saffman for treating the influence of fluid inertia, we draw the readers' attention to the work of Lovalenti & Brady (1993) who provide a thorough investigation of forces that act during the time-dependent motion of a particle in a low-Reynolds-number flow; see also Lawrence & Mei (1995).

3. Brownian motion in thin liquid films

Synopsis: Saffman's contribution to this problem was to recognize the important role that the liquid surrounding a fluid membrane has on the translational resistance experienced by particles moving along the membrane. This work was published jointly with Max Delbrück, a Caltech biologist who shared the 1969 Nobel Prize in Physiology or Medicine (for work interpreting the genetic code and its role in protein synthesis).

Biological membranes have a common structure that consists of a bilayer of lipid and protein molecules approximately 5 nm thick (Alberts *et al.* 1994, Chap. 10). In the early 1970s experimental evidence accumulated that the cell membrane had fluid-like properties (Singer & Nicholson 1972). In particular, protein molecules trapped in the membrane, and even the lipid molecules that constitute most of the membrane, were observed to diffuse along the membrane. Both experiments with synthetic membranes and actual biological membranes have given similar results and demonstrate that flow in biological membranes may be modelled, at least locally, as two-dimensional flow parallel to the membrane. It is also recognized that the fluidity is biologically important (Alberts *et al.*, p. 480). An interesting question then is to determine the lateral diffusion coefficient and how it depends on the size of the diffusing particle, the viscosity of the membrane, the viscosity of the surrounding phases, the concentration of membrane-trapped particles, and the presence of nearby boundaries.

3.1. *The hydrodynamic model*

The Stokes–Einstein equation, established on the basis of standard thermodynamic arguments (e.g. Batchelor 1976), equates the (scalar) translational diffusivity D to the ratio of the thermal energy $k_B T$, where k_B is Boltzmann's constant and T is the absolute temperature, to the particle's hydrodynamic resistance, $\zeta = F/U$, where F is the force acting on a particle translating with velocity U:

$$D = \frac{k_B T}{F/U}. \tag{8}$$

The biophysically relevant question is to establish the form of the resistance coefficient ζ for a membrane-bound particle.

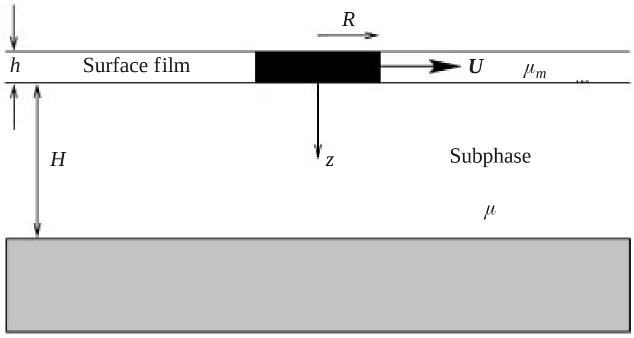

FIGURE 2. A hydrodynamic model of a membrane-bound object, modelled as a rigid cylinder, translating in a fluid surface film above a (Newtonian) subphase of finite depth.

Saffman & Delbrück (1975) and Saffman (1976) considered this problem theoretically for the case of a membrane-trapped particle, modelled as a circular cylinder spanning the membrane (figure 2). The membrane was treated as an infinite thin planar layer of viscous fluid surrounded on each side by a fluid of lower viscosity (water). There are, however, circumstances where natural membranes are surrounded by more complicated structures, which may be more viscous than the lipid bilayer (e.g. Vaz *et al.* 1987).

Motion occurs in the low-Reynolds-number flow limit and so the problem at first sight is simply to determine the resistance coefficient (in this case the force/length/velocity) for a cylinder translating in a flow where inertia is neglected. This two-dimensional flow problem, though, has no solution owing to a disturbance velocity that grows logarithmically with distance from the cylinder (e.g. Leal 1992, p. 492) – the well-known Stokes's paradox. The conundrum is usually resolved by accounting either for boundaries at a finite distance, or the presence of inertia in the fluid, which is important at large distances, or for the finite size (aspect ratio) of the cylinder. In Saffman's own words (1976, p. 594):

> To obtain a finite mobility so that the Einstein relation [$D = k_B T/\zeta$] can be employed, we can either introduce the convective acceleration terms (i.e. use the Oseen drag), or realize that the membrane is of finite size and obtain a mobility which varies inversely with the logarithm of the sheet radius, or take account of the viscosity of the aqueous fluid outside the sheet. The first approach is not viable because it is not enough for the mobility to be finite: it must be constant, independent of the velocity, for the Einstein formula. The second approach is simple ... The third appears, on examination of the results and insertion of typical values for the parameters, to be most relevant.

Thus, the important observation put forth was the recognition of the hydrodynamic importance of the surrounding fluid. Saffman (1976) showed that the logarithmic divergence characteristic of the two-dimensional flow due to a translating cylinder can be 'cut-off' by accounting for viscous resistance from the surrounding fluid and that this physical effect is expected to be more significant than either the presence of boundaries at large distances or the effect of inertia. Below I outline the idea of Saffman's (1976) calculation which makes use of the ideas of matched asymptotic expansions to treat an analytical representation of the force/velocity relation derived initially as a system of dual integral equations. Experimental evidence for the Saffman–Delbrück picture of protein and lipid diffusion along membranes is discussed by Peters & Cherry (1982), Clegg & Vaz (1985) and Vaz *et al.* (1987). The hydrodynamic model is also useful for describing the diffusion of small particles trapped in soap films (Cheung *et al.* 1996).

3.2. *The mathematical description*

The mathematical model is shown in figure 2. The membrane is modelled as a constant-viscosity Newtonian fluid of viscosity μ_m with thickness h and the surrounding fluid phases have viscosity μ. The diffusing particle is modelled as a cylinder of radius R which spans the membrane and translates with velocity $\boldsymbol{U}$. In particular, Saffman & Delbrück were concerned with the limit of an infinite fluid phase on either side of the membrane. I shall first describe their results and then consider the influence of a finite subphase of depth H; in both cases note that mathematical formulae given here account for the effect of fluid on one side only.

Detailed mathematical arguments are given in Saffman (1976; see also Stone & Ajdari 1998). The basic ideas are straightforward to describe and are based on the assumption that a dimensionless viscosity ratio $\Lambda = R\mu/h\mu_m \ll 1$ for proteins ($R \approx h$ and $\mu_m \approx 100\mu$ are common). If one considers the purely two-dimensional problem, then the disturbance velocity $u_m(r)$ in the membrane at distance r from a cylinder of radius R that experiences a force/length, F/h, is

$$u_m(r) \approx \frac{F/h}{\mu_m} \ln(r/R). \tag{9}$$

This logarithmic divergence is common in two-dimensional problems and is the mathematical origin of Stokes's paradox for a translating cylinder of infinite length. The drag exerted on the membrane by the surrounding fluid phase is not completely negligible however, and is important on a length scale ℓ_m that may be estimated by accounting for viscous forces per unit volume exerted on the sheet:

$$O\left(\frac{\mu_m u}{\ell_m^2}\right) = O\left(\frac{\mu u}{h\ell_m}\right) \qquad \Rightarrow \qquad \ell_m = O\left(\frac{\mu_m h}{\mu}\right), \tag{10}$$

where the estimate follows by recognizing that the surrounding fluid exerts a stress $O(\mu u/\ell_m)$ that is assumed to be distributed over the membrane thickness h. In the limit $\Lambda = R\mu/h\mu_m \ll 1$, the force on the cylinder is dominated by velocity gradients that occur in the membrane and these velocity variations thus occur over a length scale $O(R\ln(\ell_m/R))$. Hence, the order of magnitude of the force on the cylinder follows from

$$F = O\left(\frac{\mu_m U}{R\ln(\ell_m/R)} 2\pi Rh\right) = O\left(\frac{2\pi\mu_m hU}{\ln(\ell_m/R)}\right) = O\left(\frac{2\pi\mu RU}{\Lambda\ln(\ell_m/R)}\right), \tag{11}$$

since $\Lambda^{-1} = \ell_m/R$. This estimate is expected to resolve Stokes's paradox, as contrasted say with the influence of boundaries at a finite distance, provided ℓ_m is smaller than the finite size of the membrane, as it is in most practical cases ($\ell_m \approx 100R$ which is generally much smaller than the typical radius of curvature or size of the membrane). Saffman's exact calculation via a singular perturbation method applied to a dual integral representation of the detailed velocity field yielded

$$\boldsymbol{F}_{\text{Saffman}} = -\frac{4\pi\eta R\boldsymbol{U}}{\Lambda\left[\ln(2/\Lambda) - \gamma\right]} \quad (\Lambda \ll 1), \tag{12}$$

where $\gamma \approx 0.5772$ is Euler's constant (the force from fluid on only one side of the membrane is included). In addition, Saffman calculated the torque required to rotate a membrane-trapped particle, but this was straightforward as no divergences are encountered. Some experimental evidence for the reasonableness of the Saffman–Delbrück model was mentioned at the end of § 3.1.

Research since Saffman's papers: The problem was studied further by Hughes, Pailthorpe & White (1981) who developed an improved asymptotic expansion of the force–velocity relation as a function of the viscosity ratio parameter Λ:

$$\boldsymbol{F}_{\text{Hughes}} = -\frac{4\pi\eta R\boldsymbol{U}}{\Lambda\left[\ln(2/\Lambda) - \gamma + (4/\pi)\Lambda - \frac{1}{2}\Lambda^2\ln(2/\Lambda)\right]} \quad (\Lambda < 1). \tag{13}$$

These authors showed that (13) is in excellent agreement with numerical solutions of the original integral equation solution for $\Lambda < 0.6$.

An alternative study of the resistance offered by a thin layer of surrounding fluid was done by Evans & Sackmann (1988), who were motivated by scientific applications where artificial or biological membranes are coupled directly to a rigid substrate (e.g. Sackmann 1996). In this case a membrane-trapped particle experiences an increased viscous force from the subphase owing to the stronger frictional effects of the thin sublayer liquid (depth H). Evans & Sackmann found that the force acting on the translating particle is

$$\boldsymbol{F}_{\text{E–S}} = -\frac{4\pi\eta R\boldsymbol{U}}{\Lambda}\left[\frac{1}{2}\epsilon^2 + \frac{\epsilon K_1(\epsilon)}{K_0(\epsilon)}\right], \quad \epsilon^2 = \Lambda\frac{R}{H}, \tag{14}$$

where the $K_n(s)$ are modified Bessel functions.

Finally, Stone & Ajdari (1998) studied the coupling of motion of a membrane-trapped particle with a finite-depth subphase (not necessarily thin). For the limit $\Lambda = R\mu/h\mu_m \ll 1$, the logarithmically growing disturbance velocity (9) may be cut-off by viscous stresses from the subphase (effectively from a sublayer shear flow) on a length scale ℓ_H intermediate between R and ℓ_m, where

$$O\left(\frac{\mu_m u}{\ell_H^2}\right) = O\left(\frac{\mu u}{hH}\right) \qquad \Rightarrow \qquad \ell_H = O\left(\frac{\mu_m hH}{\mu}\right)^{1/2}. \tag{15}$$

The drag force is then expected to have magnitude

$$F = O\left(\frac{\mu_m U}{R\ln(\ell_H/R)}2\pi Rh\right) = O\left(\frac{2\pi\mu_m hU}{\ln(\ell_H/R)}\right), \tag{16}$$

and a detailed calculation gives

$$\boldsymbol{F}_{\text{finite–depth}} = -\frac{4\pi\mu R\boldsymbol{U}}{\Lambda\left[\ln(\Lambda R/4H)^{-1/2} - \gamma\right]}. \tag{17}$$

The above results characterize the resistance offered by the subphase fluid. For the limit $\Lambda \ll 1$, we expect the original Saffman–Delbrück formula (12) to be useful when $R \ll \ell_m < \ell_H$, while the finite-depth equivalent (17) is a good approximation to the force–velocity relation when $R \ll \ell_H < \ell_m$. The Evans–Sackmann thin-film formula (14) is useful provided the film is sufficiently thin, which requires $H < \ell_m$.

The Saffman–Delbrück approach is also the starting point for other investigations concerned with membrane transport. For example, Koch, Hammer and coworkers have explored the influence on diffusion of a finite concentration of protein particles using both analytical as well as numerical methods (e.g. Bussell, Hammer & Koch 1994; Dodd *et al.* 1995). These authors have shown that accounting for such finite concentrations can explain the reduced diffusivities that are often measured. Furthermore, they point out that in plasma membranes some of the proteins are actually fixed, so that the proteins free to diffuse see a medium more akin to a fixed bed, which more effectively eliminates long-range motions (Brinkman screening). In another line of research, experiments with both biological and artificial lipid monolayers that exist in the two-phase regime demonstrate that a variety of two-dimensional patterns can be formed (e.g. McConnell 1991). The dynamics of these systems have been modelled using a combination of electrostatic driving forces characteristic of the dipolar molecules that make up the membrane and hydrodynamic resistance (predominantly from the subphase) that is modelled using the Saffman–Delbrück approach (e.g. Stone & McConnell 1995; De Koker 1996). Finally, a recent investigation has measured the force accompanying controlled translation of a membrane-trapped disk, and, using an analytical solution for the thin subphase limit, determines the viscosity of a surfactant monolayer (Barentin *et al.* 1999).

4. Particle motion in rotating viscous flows

Synopsis: In joint work with Derek Moore, a mathematician at Imperial College, Saffman investigated the structure of viscous boundary layers which are crucially important when particles move in rapidly rotating fluids. The theoretical predictions have been shown to be in good agreement with experiments, and include some rather surprising flow features.

Rotating fluid motions occur in industrial devices such as centrifuges or in localized vortical regions of flow. The distribution of particles in such flows is determined by the hydrodynamic and external forces acting and, as the background solid-body rotation changes the local flow field around suspended particles, the hydrodynamic forces are in fact different than just a simple Stokes drag. For example, when an object translates relative to a fluid rotating with angular velocity Ω it tends to drag along with it a column of fluid that lies parallel to the rotation axis. This rather unexpected effect was first investigated experimentally by Taylor (1922), who studied motions both parallel and transverse to the rotation axis. The basic mathematical reason for the existence of such columnar regions of fluid, the so-called Taylor columns, can be traced to the Taylor–Proudman theorem (e.g. Greenspan 1968), which states that for steady motions in a rotating flow in which viscous and inertial influences are negligible $\Omega \cdot \nabla \boldsymbol{u} = \boldsymbol{0}$, i.e. the velocity does not vary along the direction of the rotation axis.

Most of the theoretical modelling of particle motion in rotating flows begins with the Navier–Stokes equation written for a coordinate system rotating steadily with angular velocity Ω (Batchelor 1967, p. 162):

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + 2\Omega \wedge \boldsymbol{u} = -\frac{1}{\rho}\nabla p_d + \nu \nabla^2 \boldsymbol{u}, \tag{18}$$

where p_d is the dynamic pressure incorporating gravitational and centrifugal effects. For steady motions in the limit that convective inertial effects are small (the low-Rossby-number limit, $U/\Omega a \ll 1$) we arrive at the dimensionless equations for rotating viscous flows:

$$2\mathscr{T} \boldsymbol{e}_z \wedge \boldsymbol{u} = -\nabla p_d + \nabla^2 \boldsymbol{u}, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{19}$$

where lengths have been scaled by the particle radius a, velocities by the translation speed U, pressure by $\mu U/a$, and the Taylor number $\mathscr{T}$ is defined as

$$\mathscr{T} = \frac{\Omega a^2}{\nu}. \tag{20}$$

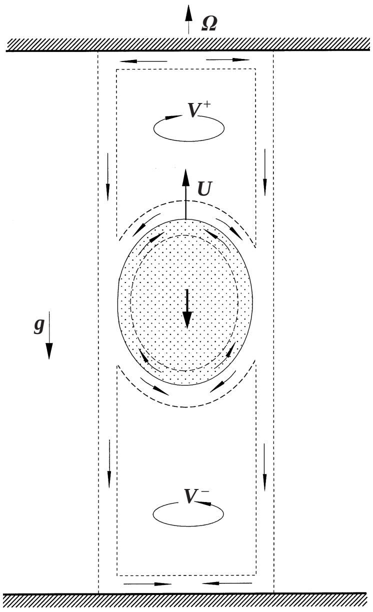

FIGURE 3. A schematic illustration of the flow induced by a low-viscosity drop rising at velocity $\boldsymbol{U}$ between rigid horizontal boundaries rotating at angular velocity Ω. The drop interface is characterized by a double Ekman layer, and the internal flow by a weak downward flow (Bush *et al.* 1995). The swirl velocities fore and aft of the rising drop are indicated also.

Equation (19), a linearized version of the Navier–Stokes equation, was apparently first applied to particle motions by Morrison & Morgan (1956).

4.1. *Axial particle motion*

The flow field and forces accompanying steady particle motion in the low-Rossby-number, high-Taylor-number limit were considered theoretically in a series of papers by Moore & Saffman (1968, 1969*a, b*). First, Moore & Saffman studied the on-axis rise of a particle with speed U in a fluid bounded above and below by rigid horizontal surfaces. The theoretical description was predicated on the existence of a Taylor column spanning the entire distance between the horizontal boundaries (figure 3). Their paper provides a clear physical picture showing the origin of the drag as arising from a pressure difference generated from a difference in swirl velocities that exist fore and aft of the translating particle. In addition, there are viscous boundary layers along the particle surface and the horizontal boundaries. A sketch of the typical fluid motions to be expected in the high-Taylor-number, bounded-Taylor-column limit is given in figure 3 for the case of a drop of liquid rising through a rapidly rotating fluid (Bush, Stone & Bloxham 1995); note the swirls produced ahead of (behind) the particle by shortening (lengthening) the vortex lines in the main body of the Taylor column and the existence of boundary layers exterior and interior to the drop surface.

The physical argument given by Moore & Saffman (1968) is very instructive for understanding the magnitude of the high-Taylor-number ($\mathscr{T} \gg 1$), pressure-dominated drag force. We will discuss this limit with reference to (19). In particular, on the top (bottom) rigid boundary there is a viscous boundary layer with thickness $\delta \propto a\mathscr{T}^{-1/2}$, and the flow into this boundary layer from the main column of fluid associated with the Taylor column and the rising particle induces a swirl velocity of magnitude $O(U\mathscr{T}^{1/2})$ that is transmitted, according to the Taylor–Proudman theorem, to the entire column above (below) the particle. According to (19) the swirls require a variation of pressure of magnitude $O((\mu U/a)\mathscr{T}^{3/2})$ and consequently the force has magnitude $O(\mu Ua\mathscr{T}^{3/2})$, which is clearly much larger than the corresponding Stokes drag. Moore & Saffman then provide the analytical details for treating the Ekman (boundary) layers along the surfaces, and coupling them via mass conservation, to arrive at a complete analytical description of the flow.

Research since the Moore & Saffman papers: A complete picture of the flow requires fluid to return from the fore to the aft of the particle – the usual picture of this transport is via narrow Stewartson layers, along the sides of the Taylor column, as characterized analytically by Moore & Saffman (1969*a*). An important assumption in the above description of the bounded flow is that the layer depth H exceeds the boundary layer thickness, $H \gg a\mathscr{T}^{-1/2}$, and further that the Taylor column actually spans the entire depth of the bounded fluid region, which requires $H \ll a\mathscr{T}^{1/2}$. This latter assumption was relaxed by Hocking, Moore & Walton (1979) and discussed in more detail by Ungarish & Vedensky (1995) who succeeded in presenting a solution to (19) for arbitrary Taylor numbers for a thin circular disk midway between the boundaries. The case of axial motion in an unbounded fluid was treated in the low-Taylor-number limit by Childress (1964; spheres) and studies allowing for arbitrary Taylor numbers are reported by Vedensky & Ungarish (1994; thin disks) and Tanzosh & Stone (1994; spheres).

Experiments by Maxworthy (1968) exhibited some discrepancies with the Moore & Saffman 'short-container' theory, though the trends were all correct. The calculations by Hocking *et al.* (1979) did not provide an improved comparison with experiment. Additional experiments were conducted by Bush *et al.* (1995) with droplets and discrepancies were also noted when the experimental results were compared with an appropriate generalization of the Moore–Saffman theory. Ungarish (1996) has further investigated the discrepancies between theory and experiment and argues that the differences may not be due to the neglect of inertial influences in the theory, but rather are probably due to viscous influences not captured by the asymptotic ($\mathscr{T} \gg 1$) boundary layer ideas.

An interesting application of axial particle motion in rapidly rotating flows, as described by the theory of Moore & Saffman, is the so-called spinning drop electrophorometer (apparently developed originally in the 1920s). In this device, application of an electric field along the axis of a cylindrical tube causes translation of a small suspended bubble (electrophoresis) and rapid rotation of the cylindrical tube is used to maintain the bubble along the centreline. The measured translation speed of the bubble, coupled with the force/velocity relation in a rotating system as described in the papers mentioned above, is used to deduce the surface zeta potential. A hydrodynamic description of this process is given by Sherwood (1986); see also Graciaa *et al.* (1995).

4.2. *Transverse particle motion*

The case of transverse particle motion according to (19) is in some ways considerably more difficult to quantify. The flow field was discussed by Moore & Saffman (1969*b*) in the high-Taylor-number limit for transverse translation of a thin disk in a bounded container such that the Taylor column again spans the fluid layer. They obtained the surprising predictions that fluid particles can cross the Taylor column associated with thin objects (i.e. the Taylor column is not a stream surface as originally identified by Taylor (1922) in experiments with transverse translation of spheres) and the fluid particle paths are bent through an angle of approximately 18° as they pass over the disk. Moore & Saffman also compared their results to the more familiar case of 'fat' bodies, i.e. those whose thickness is comparable to the Ekman layer thickness. In an appendix to the paper, Maxworthy provided an experimental verification that fluid particles do indeed pass over the transversely moving particle where they typically are rotated through an angle of 20°, which is in excellent agreement with the theoretical predictions.

Research since the Moore & Saffman paper: Additional theoretical studies of (19) for transverse particle motion in unbounded flows include the translation of a sphere in the low-Taylor-number limit (Herron, Davis & Bretherton 1975) and the translation of a thin disk for arbitrary Taylor numbers (Tanzosh & Stone 1995); Moore & Saffman (1969*b*) treat the high-Taylor-number limit. It is interesting to note that there are well-known analogies between the governing equations for rotating and stratified flows (e.g. Veronis 1970), and Foster & Saffman (1970) considered the transverse motion of a particle in a stratified medium. However, the theoretical structure of the problem for particle translation does not have many similarities with the Moore–Saffman description outlined above owing to the different form of boundary conditions imposed on the density distribution.

The reader interested in a discussion of rotating suspension flows (e.g. centrifugal separation), which utilizes some of the fundamental results from studies of isolated particles in rotating viscous flows, is referred to Ungarish (1993). For a recent review of the literature on particle motion in rotating flows see Bush, Stone & Tanzosh (1994).

5. Additional contributions

It is impossible to provide even a superficial description of the many other contributions Saffman made to fluid dynamics and transport processes. Here it seems wise to indicate three additional contributions, which again serve to illustrate the breadth and importance of his scientific research: (1) dispersion in porous materials, (2) suspension mechanics with a focus on sedimentation, and (3) compressible, low-Reynolds-number flows.

5.1. *Dispersion in porous media*

Flow in porous materials occurs in a wide variety of industrial and man-made settings. The dispersal of a chemical is an important transport process and it is now well appreciated that the spreading, or dispersion, of an injected tracer, naturally occurring chemical, or pollutant in porous materials depends on the average velocity (U) through the medium and the typical pore size (ℓ) of the medium. One of the first studies to quantify the lateral and longitudinal dispersion processes in a porous medium was given by Saffman (1959, 1960), who recognized the important role played by molecular diffusion (D_m), even when its magnitude is small; a closely related independent investigation was reported by Josselin de Jong (1958).

Saffman's papers on dispersion use a simplified description of the actual transport process in the form of a model based upon a network of capillary tubes and focus on the stochastic element of the transport inevitably introduced by the complex underlying microstructure (so-called mechanical dispersion). The important dimensionless parameter is the Péclet number, defined as $\mathscr{P} = U\ell/D_m$. The 1959 paper treats the porous medium as an assembly of randomly oriented straight uniform pores, introduces a statistical description of the transport process in terms of a random walk, with molecular diffusion entering as a parameter affecting the time step in the random walk, and treats $\mathscr{P} \gg 1$. The 1960 paper considers $\mathscr{P} = O(1)$. Saffman anticipates the most important physical processes and the quantitative manner in which they affect spreading of a solute, which is characterized by lateral ($\mathscr{D}_\perp$) and longitudinal ($\mathscr{D}_\parallel$) dispersivities. In particular, as most transport processes have $\mathscr{P} \gg 1$, Saffman demonstrates that for lateral dispersion $\mathscr{D}_\perp/D_m = O(\mathscr{P})$, while the magnitude of the longitudinal dispersivity scales as $\mathscr{D}_\parallel/D_m = O(\mathscr{P} \ln \mathscr{P})$.

In fact, it is now recognized (e.g. Koch & Brady 1985) that it is useful to treat the dispersion process in terms of (*a*) mechanical dispersion, where $\mathscr{D}/D_m = O(\mathscr{P})$, (*b*) holdup dispersion, due to regions of trapped flow or closed streamlines, for which $\mathscr{D}/D_m = O(\mathscr{P}^2)$ analogous to the familiar Taylor–Aris dispersion result for the spreading of a pulse in a laminar, parabolic tube or channel flow, and (*c*) boundary layer dispersion, for which $\mathscr{D}/D_m = O(\mathscr{P} \ln \mathscr{P})$, as recognized by Saffman and Josselin de Jong. This last result may be best explained as a consequence of the interplay of convection and diffusion in a thin boundary layer of thickness $\delta \propto \ell \mathscr{P}^{-1/3}$ adjacent to rigid boundaries (Koch & Brady 1985). For an excellent discussion of both experiments on and theory of dispersion in porous materials, the reader is referred to a multi-author section on dispersion in Guyon, Nadal & Pomeau (1988).

In effect, Saffman was among the first researchers to clearly explain the common result that the mechanical dispersivity $\mathscr{D} \propto U\ell$ results from spreading due to the stochastic nature of the velocity (produced by the complex microstructure) and hence is a transport mechanism independent of molecular diffusion. Also, the possibility of high-Péclet-number dispersion scaling as $\mathscr{P} \ln \mathscr{P}$, which is in good agreement with the data (e.g. Koch & Brady 1985), can be traced to Saffman's first paper in this field.

5.2. *Settling speed of suspensions*

A long-standing problem in suspension mechanics is to predict the rate of sedimentation of a random suspension of rigid, spherical particles. An isolated sphere with density ρ_p sediments with its Stokes fall velocity $\boldsymbol{U}_s = 2(\rho_p - \rho)a^2\boldsymbol{g}/(9\mu)$. However, the average fall velocity $\boldsymbol{U}_{sed}$ in a suspension of volume fraction ϕ is reduced from its Stokes value, largely due to the back-flow that accompanies sedimentation; Batchelor (1972) finds $\boldsymbol{U}_{sed} = \boldsymbol{U}_s(1 - 6.55\phi)$ for $\phi \ll 1$. The seemingly straightforward calculation of the first correction to the sedimentation velocity is complicated enormously by the long-range character of disturbance velocities in Stokes flows. Two closely related flow problems are the sedimentation rate of a regular array of spheres (the particles experience the same force and velocity, in contrast to a sedimenting random suspension where the forces on each particle are the same but the velocities differ) and the translation of, or the flow through, a random array of fixed spheres (velocities specified but the forces on the particles differ). Saffman demonstrated the precise reason why these three problems, despite seeming to be very similar, can give different scaling laws for the change in the mean velocity as a function of the volume fraction of suspended particles. The reader may note that a fourth problem in this class is the average stress in a two-phase suspension (e.g. Batchelor 1976; Hinch 1977), while a fifth is the variance of the sedimentation velocity of a suspension (Caflisch & Luke 1985; Koch & Shaqfeh 1991). This latter problem has attracted much recent attention owing to experimental results that are in apparent disagreement with the theoretical predictions (Segrè, Herbolzheimer & Chaiken 1997; see also Brenner 1999).

Saffman (1973) provided a useful method for thinking about this class of problems by making the point-particle approximation, where the particles are replaced by a multipole distribution of forces (force, torque, stresslet) at their centres. For the case of sedimentation only the monopole or Stokeslet is needed. The basic question concerns the concentration dependence in the relation of the average force on the particles to the average velocity in the system. Saffman's paper clearly demonstrates the manner in which the particle distribution (periodic array or random) and the application of forces (identical forces for sedimentation but a distribution of forces for a uniformly translating, or flow through, a fixed random array) leads to different first corrections as functions of ϕ to the average velocity. In particular, for a regular periodic array of particles the change in the sedimentation velocity is $O(\phi^{1/3})$, while for a random distribution of sedimenting particles (free to move relative to one another) the change in the sedimentation speed is $O(\phi)$. In the case of translation of, or flow through, a fixed random array (which serves as a model of a porous media) the change in the average velocity is proportional to $O(\phi^{1/2})$, i.e. Brinkman screening.

It is now possible to do complete computer simulations for these suspension flow problems (e.g. Brady & Bossis 1988). Nonetheless, Saffman's paper introduces a common approach to these seemingly different problems, and will likely remain a useful educational medium for future researchers.

5.3. *Compressibility effects at low Reynolds numbers*

As an additional area of research to include in this discussion, we note that Saffman collaborated on two papers studying the effect of compressibility in lubrication configurations. In particular, Taylor & Saffman (1957) investigated air flow in the narrow gap between a rotating disk and a rigid plane. The analysis modifies the usual Reynolds lubrication equation to account for compressibility of the lubricant and two aspects of the experimental configuration are studied in detail: (*a*) time-periodic oscillations of the spinning disk in a direction parallel to the rotation axis and (*b*) slightly non-parallel surfaces. The results of the analysis provide some rational explanation for experimental measurements earlier ascribed to non-Newtonian properties of air. Also, numerical solutions (apparently some of the first) are given for the effect of compressibility on the pressure distribution in the narrow gas film between rotating and fixed inclined surfaces. Later, a more detailed numerical and asymptotic investigation of compressibility effects was made of this latter problem (Cole, Keller & Saffman 1967). These studies, and in particular the 1957 paper, remain of value today since compressibility (and slip effects) during gas flows in narrow geometries are common in microelectromechanical (MEMS) devices (e.g. Harley *et al.* 1995) and air bearing sliders used in computer hard disk drives (Witelski 1998).

This article is dedicated to Professor Philip G. Saffman, whose scientific papers have contributed immensely to the author's education in fluid dynamics. The breadth and depth of Professor Saffman's scientific insights, as well as his style of combining physical insights and mathematical arguments and analyses, have often served to inspire the author, who recognizes the immense contribution Saffman has made to the fluid dynamics, mathematics and physics communities.

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